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INDECOMPOSABLE LIE ALGEBRAS WITH NONTRIVIAL LEVI DECOMPOSITION CANNOT HAVE FILIFORM RADICAL

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Abstract

It is shown that a semidirect sum $\mathfrak{g} = \mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ of a semisimple Lie algebra $\mathfrak{s}$ and a solvable Lie algebra $\mathfrak{r}$ with respect to a representation of $\mathfrak{s}$ which does not decompose into a direct sum of ideals cannot have a radical $\mathfrak{r}$ associated to a filiform Lie algebra. This proves that this class of nilpotent Lie algebras has none interest for the structure theory of nonsolvable Lie algebras.

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1 Introduction

By means of the Levi decomposition theorem, the structure theory of Lie algebras reduces essentially to the analysis of semisimple and solvable Lie algebras, the first class having been classified long ago and constituting nowadays a classical result. For solvable Lie algebras no structure theory like in the semisimple case exists, and the absence of determining invariants like nondegenerate trace forms or root systems prevents from a global classification. Only in dimensions $n \leq 6$ satisfactory results exist, mainly due to G. M. Mubarakzyanov [9, 10, 11], and completed more recently by P. Turkowski [13]. This is particular allowed to determine the isomorphism class of nonsolvable Lie algebras having a nontrivial Levi decomposition up to dimension nine [12], as well as partial results in higher dimensions. Lie algebras having a nontrivial Levi decomposition are of actual interest, not only for questions related to their representations and structure, but also for some of their physical applications [2, 14].
Within the class of solvable Lie algebras, nilpotent algebras have been devoted a great number of works in the last decades, mainly centered on two types of algebras, the most nilpotent (also called metabelian, and which are of great interest for differential geometric applications [4, 5], and the less nilpotent ones (also called filiform), and which have played an important role in rigidity theory [1, 3]. The knowledge on this class is almost absolute, since all relevant facts as their derivations, weight systems, associated gradings or completability properties are well known (see e.g. [1] and references therein).

In this paper we show that, although filiform Lie algebras are of interest for the study of solvable Lie algebras, this does not extend to the class of nonsolvable algebras. More precisely, we prove that the only Lie algebras in dimension ≥ 7 with nontrivial Levi part and filiform radical are necessarily copies of the trivial representation of $\mathfrak{s}$, thus direct sums of algebras [1]. This implies that filiform are of none interest for the structural theory of nonsolvable algebras.

Unless otherwise stated, any Lie algebra $\mathfrak{g}$ considered in this work is indecomposable (i.e., does not reduce to a direct sum of ideals) and is defined over the field $\mathbb{C}$ of complex numbers. We convene that nonwritten brackets are either zero or obtained by antisymmetry.

[1] There is of course the trivial example of the six dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_{D_{\frac{1}{2}}\oplus D_0}\mathfrak{h}_1$, but this particular case is due to the fact that the Heisenberg algebra is filiform in dimension three.

2 Preliminaries and notation

The Levi decomposition theorem states that any Lie algebra is formed as a semidirect product of a semisimple Lie algebra $\mathfrak{s}$, called the Levi factor of $\mathfrak{g}$, and a maximal solvable ideal $\mathfrak{r}$, called the radical. Since the latter is an ideal, the Levi factor $\mathfrak{s}$ acts on $\mathfrak{r}$, and there are two possibilities for this action:

$$[\mathfrak{s}, \mathfrak{r}] = 0$$
$$[\mathfrak{s}, \mathfrak{r}] \neq 0$$

If the first holds, then $\mathfrak{g}$ is a direct sum $\mathfrak{s} \oplus \mathfrak{r}$, whereas the second possibility implies the existence of a representation R of $\mathfrak{s}$ which describes the action, i.e.,

$$[x, y] = R(x) .y, \ \forall x \in \mathfrak{s}, y \in \mathfrak{r} \tag{1}$$

Unless there is no ambiguity, it is more convenient to write $\overrightarrow{\oplus}_R$ instead of $\overrightarrow{\oplus}$, which is the common symbol for denoting semidirect products. Since (1) implies that the radical is a module over $\mathfrak{s}$, we have to expect severe restrictions on the structure of the radical, while for direct sums any solvable Lie algebra is suitable as radical. Indeed, $\mathfrak{r}$ is a nilpotent algebra if the representation R does not posses a copy of the trivial representation D_0 of $\mathfrak{s}$ [12].

Recall that given a nilpotent Lie algebra $\mathfrak{n}$ we can associate to it the following sequence, called central descending sequence:

$$C^0(\mathfrak{n}) = \mathfrak{n}, \quad C^i(\mathfrak{n}) = [C^{i-1}(\mathfrak{n}), \mathfrak{n}], \quad i \geq 1.$$

An n-dimensional nilpotent Lie algebra $\mathfrak{n}$ is called filiform if

$$\dim C^i(\mathfrak{n}) = n - 1 - k, \quad k \geq 1.$$

There is a distinguished filiform algebra, called L_n, and which suffices to obtain all other algebras of this type. Over a special basis $\{Y_1, .., Y_n\}$ its brackets adopt the form:

$$[Y_1, Y_i] = Y_{i+1}, \quad 2 \leq i \leq n - 1$$

If $Der(\mathfrak{n})$ denotes the algebra of derivations, a torus T over $\mathfrak{n}$ is an abelian subalgebra of $Der(\mathfrak{n})$ consisting of semi-simple endomorphisms. The torus T induces a natural representation on the Lie algebra $\mathfrak{n}$, such that we obtain the following decomposition:

$$\mathfrak{n} = \sum_{\alpha \in T^*} \mathfrak{n}_\alpha \tag{2}$$

where $T^* = Hom_{\mathbb{C}}(T, \mathbb{C})$ and $\mathfrak{n}_\alpha = \{X \in \mathfrak{n} \mid [t, X] = \alpha(t) X \ \ \forall t \in T\}$ is the weight space corresponding to the weight α. If the torus is maximal for the inclusion relation, as tori are conjugated, its common dimension is a numerical invariant of $\mathfrak{n}$ called the rank and denoted by $r(\mathfrak{n})$. Following [6], we call

$$R\mathfrak{n}(T) = \{\alpha \in T^* \mid \mathfrak{g}_\alpha \neq 0\} \tag{3}$$

the set of weights for the representation of T over $\mathfrak{g}$ and

$$P\mathfrak{n}(T) = \{(\alpha, d\alpha) \mid \alpha \in R\mathfrak{n}(T),\ d\alpha = \dim \mathfrak{n}_\alpha\} \tag{4}$$

the weight system of $\mathfrak{n}$ (with respect to T). The equivalence class of a weight system over a Lie algebra $\mathfrak{n}$ is an invariant of $\mathfrak{n}$ [6].

3 Non-filiformity of radicals

Let $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}$ be an indecomposable Lie algebra. The objective of this note is to prove the following

Theorem 1. *Let $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}$ be an indecomposable Lie algebra with nontrivial Levi decomposition. Then its radical $\mathfrak{r}$ cannot be a filiform Lie algebra.*

By means of reduction we will see that it suffices to prove the assertion for the case of a rank one Levi subalgebra.

Lemma 1. *Any Lie algebra $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}$ with filiform radical $\mathfrak{r}$ contracts onto the Lie algebra $\mathfrak{g}' = \mathfrak{s} \overrightarrow{\oplus}_R L_n$.*

Proof. Let $\{X_1, .., X_m, X_{m+1}, .., X_{m+n}\}$ be a basis of $\mathfrak{g}$, where $\{X_1, .., X_m\}$ is a basis of the Levi part and $\{X_{m+1}, .., X_{n+m}\}$ is a basis of $\mathfrak{r}$. Let $\left\{C_{ij}^k\right\}_{1 \leq i,j,k \leq n+m}$ be the structure constants of $\mathfrak{g}$. Since the radical is filiform, without loss of generality we can suppose that $\{X_{m+1}, .., X_{n+m}\}$ is arranged in such manner that

$$[X_{m+1}, X_{m+1+j}] = \lambda_{m+1+j} X_{m+2+j},\ 1 \leq j \leq n-2 \tag{5}$$

holds, where $\lambda_{m+1+j} \neq 0$ for any j. Consider the change of basis:

$$\begin{aligned} X_i' &= X_i,\ 1 \leq i \leq m+1 \\ X_i' &= \frac{1}{\epsilon} X_i,\ m+2 \leq i \leq n+m \end{aligned}$$

With respect to this transformation, the brackets of $\mathfrak{g}$ are altered as follows:

$$\begin{aligned} \left[X_i', X_j'\right] &= [X_i, X_j],\ 1 \leq i < j \leq m+1 \\ \left[X_i', X_{m+j}'\right] &= \frac{1}{\epsilon}[X_i, X_{m+j}],\ 1 \leq i \leq n+1,\ 2 \leq j \leq n \\ \left[X_{m+i}', X_{m+j}'\right] &= \frac{1}{\epsilon^2}\left[X_{m+i}', X_{m+j}'\right],\ 2 \leq i, j \leq n \end{aligned}$$

This shows that the brackets of the Levi part and the action of $\mathfrak{s}$ on the radical remain unchanged, while the brackets of the radical adopt the form:

$$\begin{aligned}\left[X'_{m+1}, X'_{m+i}\right] &= \lambda_{m+1+i} X_{m+1+i},\ 2 \leq i \leq n-2 \\ \left[X'_{m+i}, X'_{m+j}\right] &= \frac{1}{\epsilon} C^{m+k}_{m+i,m+j} X'_{m+k},\ 2 \leq i,j,k \leq n\end{aligned}$$

Therefore for $\epsilon \to \infty$ we obtain

$$\left[X'_{m+i}, X'_{m+j}\right] = 0,\ 2 \leq i,j,k \leq n \tag{6}$$

which shows that the Lie algebra $\mathfrak{s} \overrightarrow{\oplus}_R L_n$ is an Inönü-Wigner contraction [8] of $\mathfrak{g}$. ☐

By this result, it suffices to study the case where the radical is filiform and isomorphic to L_n, since all remaining cases can be obtained by deformation. We can further simplify the problem if we consider the branching rules for representations of semisimple Lie algebras. Since any complex semisimple Lie algebra $\mathfrak{s}$ has a subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$, the embedding $\mathfrak{sl}(2,\mathbb{C}) \hookrightarrow \mathfrak{s}$ induces

$$\mathfrak{sl}(2,\mathbb{C}) \overrightarrow{\oplus}_{\sum W_i} \mathfrak{r} \hookrightarrow \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r} \tag{7}$$

where $R = \sum W_i$ is the branching rule of R with respect to the subalgebra $\mathfrak{sl}(2,\mathbb{C})$ of $\mathfrak{s}$. It should be noted that different embeddings of $\mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{s}$ may give different branching rules. This is however irrelevant for us in here, because we are only interested in the nontriviality of the representations W_i obtained in (7).

Proposition 1. *If* $\dim \mathfrak{r} > 3$, *no indecomposable Lie algebra* $\mathfrak{sl}(2,\mathbb{C}) \overrightarrow{\oplus}_R \mathfrak{r}$ *with radical isomorphic to the nilpotent Lie algebra* L_n *exists.*

Proof. Consider the Lie algebras $\mathfrak{sl}(2,\mathbb{C}) \overrightarrow{\oplus}_R L_n$, where R is a reducible representation of $\mathfrak{sl}(2,\mathbb{C})$. Let $\{X_1, X_2, X_3, Y_1, .., Y_n\}$ be a basis of $\mathfrak{sl}(2,\mathbb{C}) \overrightarrow{\oplus}_R L_n$ such that $\{X_1, X_2, X_3\}$ spans $\mathfrak{sl}(2,\mathbb{C})$ (with the brackets $[X_2, X_3] = X_1,\ [X_1, X_i] = 2(-1)^i X_i,\ i = 2,3$) and $\{Y_1, .., Y_n\}$ is a basis of L_n. Since for the central descending sequence $C^i L_n$ we have $\dim C^1 L_1 = n-1-i$ $(1 \leq i \leq n-2)$, there always exists a pair of elements $Y \in L_n - C^1 L_n,\ Z \in C^{n-3} L_n$ such that

$$[Y, Z] = Z(L_n)$$

where $Z(L_n)$ denotes the centre of L_n. This shows that the representation R necessarily contains a copy of the trivial representation D_0 of $\mathfrak{sl}(2,\mathbb{C})$. Applying the Jacobi identity to the triple $\{X_1, Y, Z\}$ we have that

$$[Y, [X_1, Z]] = [Z, [X_1, Y]]$$

Since X_1 acts like a diagonal derivation on L_n, $[X_1, Y_i] = \lambda_i Y_i$ for any i, and in particular it follows that

$$\lambda_Y + \lambda_Z = 0. \tag{8}$$

Without loss of generality we can suppose that $Y = Y_1$ and $Y_{n-1} = Z$. The important fact is that the generator of the centre has zero weight. Now the weight system of L_n has the form:

$$\Phi = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, .., (n-2)\,\alpha_1 + \alpha_2\} \tag{9}$$

thus any diagonalizable derivation of this algebra is a combination of the derivations f_1, f_2 whose eigenvalues are, respectively:

$$\begin{array}{rcl} f_1 & = & (1, 0, 1, 2, .., n-2) \\ f_2 & = & (0, 1, 1, 1, .., 1) \end{array}$$

Equation (8) implies that $[X_1, L_n] = (af_1 + bf_2)L_n$ subjected to the condition

$$a\,(n-2) + b = 0$$

thus the eigenvalues are

$$a\,(1, (2-n)\,, .., k+2-n, .., -1, 0)\,. \tag{10}$$

If $n > 3$, the sequence (10) implies that

$$k + 2 - n < 0,\ 0 \leq k \leq n-3 \tag{11}$$

Since for any representation R of $\mathfrak{sl}\,(2,\mathbb{C})$ the number of summands in the decomposition into irreducible representations is given by $\mathrm{mult}_0\,(R) + \mathrm{mult}_1\,(R)$, $\mathrm{mult}_i\,(R)$ denoting the multiplicity of the weight i, we conclude from (11) that the only irreducible representation of $\mathfrak{sl}\,(2,\mathbb{C})$ having odd maximal weight λ appearing in the decomposition of R can be the fundamental representation V_1 of weight $\lambda = 1$. But the remaining weights have all negative value, which cannot happen in view of the structure of the irreducible representations of $\mathfrak{sl}\,(2,\mathbb{C})$. This shows that we must actually have $a = 0$ in (10), from which $R = nD_0$ follows. Therefore the Lie algebra $\mathfrak{sl}\,(2,\mathbb{C})\,\overrightarrow{\oplus}_R L_n$ is a direct sum, thus decomposable. $\square$

By proposition 1, no indecomposable Lie algebra $\mathfrak{sl}\,(2,\mathbb{C})\,\overrightarrow{\oplus}_R L_n$ can be obtained. By (7), the result extends to any semisimple Lie algebra. In fact, any nontrivial representation R of a semisimple Lie algebra $\mathfrak{s}$ will provide a nontrivial branching rule for the embedding (7), from which the decomposability of $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R L_n$ follows. The assertion is a consequence of applying lemma 1.

Theorem 1 has an immediate consequence concerning the solvable Lie algebras related to filiform algebras.

Corollary 1. *No indecomposable Lie algebra $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}$ with a radical whose nilradical is filiform exists.*

Proof. Since $\mathfrak{r}$ is solvable non-nilpotent, the representation R of $\mathfrak{s}$ necessarily contains copies of the trivial representation D_0 of $\mathfrak{s}$. Since $\mathfrak{r} = \mathfrak{n} \oplus \mathfrak{t}$ by virtue of a result of Goto [7], where $\mathfrak{t}$ is a torus of derivations and $\mathfrak{n}$ the nilradical, $\mathfrak{g}$ admits a subalgebra

$$\mathfrak{g}' = \mathfrak{s} \overrightarrow{\oplus}_{R'} \mathfrak{n} \tag{12}$$

which is decomposable by theorem 1. But then $\mathfrak{g}$ is also a direct sum, since $R|_{\mathfrak{t}} = (\dim \mathfrak{t})\, D_0$. □

Finally, from the proof of proposition 1 we also deduce the following result:

Corollary 2. *No indecomposable Lie algebra $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_R \mathfrak{r}$ with $\mathfrak{s} \neq 0$ has a characteristically nilpotent radical.*

Proof. Recall that a nilpotent Lie algebra $\mathfrak{r}$ is called characteristically nilpotent if any derivation f of $\mathfrak{r}$ is nilpotent. In view of (1), this implies that for the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}$ we have

$$[\mathfrak{h}, \mathfrak{r}] = 0. \tag{13}$$

Since for any $H_i \in \mathfrak{h}$ we can find nonzero vectors $X_i, Y_i \in \mathfrak{s}$ such that

$$[H_i, X_i] = 2X_i,\ [H_i, Y_i] = -2Y_i,\ [X_i, Y_i] = H_i,$$

from this we obtain, for any $Z \in \mathfrak{r}$:

$$[Z, [H_i, X_i]] + [X_i, [Z, H_i]] - [H_i, [X_i, Z]] = 0$$

from which we deduce

$$[X_i, Z] = 0$$

by virtue of (13). Thus $[\mathfrak{s}, \mathfrak{r}] = 0$ follows at once from the Chevalley-Serre relations of $\mathfrak{s}$. □

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